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Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order

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ABSTRACT

We investigate the existence of nontrivial solutions for a multi-point boundary value problem for fractional differential equations. Under certain growth conditions on the nonlinearity, several sufficient conditions for the existence of nontrivial solution are obtained by using Leray–Schauder nonlinear alternative. As an application, some examples to illustrate our results are given.

1. Introduction

Fractional differential equation can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetic, etc [1–3]. Recently, there are a large number of papers dealing with the solvability of nonlinear fractional differential equations; see [4–10] and the references therein. The author of [11] used the contraction mapping principle and Krasnoselskii's fixed point theorem to study existence and uniqueness results in a Banach space for a three-point boundary value problem involving a fractional differential equation given by

$$\begin{aligned}&{}_CD_{0+}^{q}x\left(t\right)=f\left(t,u\left(t\right)\right),\quad t\in\left[0,T\right],\ 0<q<1,\\&a\,x\left(0\right)+g(x)=x_0,\end{aligned}$$

where ${}_CD_{0+}^{q}$ is the Caputo fractional derivative. In [12], the authors investigated the existence and multiplicity results of positive solutions by using some fixed point theorems for the three-point boundary value problem involving the Riemann–Liouville fractional differential equation given by

$$\begin{aligned}&{}_RD_{0+}^{\alpha}u\left(t\right)+f\left(t,u\left(t\right)\right)0,\quad t\in\left(0,T\right),\ 1<\alpha\leq 2,\\&u\left(0\right)=0,\quad {}_RD_{0+}^{\beta}u\left(1\right)=a_RD_{0+}^{\beta}u\left(\eta\right),0<\beta\leq 1.\end{aligned}$$

Multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [13,14]. Since then, nonlinear second-order multi-point boundary value

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problems have been studied by several authors. Recently, in [15] the authors have studied the existence of multiple positive solutions for the nth-order m-point boundary value problem

$$u^{(n)}(t) + f(t, u(t)) = 0, \quad t \in [0, 1],$$
$$u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i),$$

where n is an integer.

The purpose of this paper is to establish the existence of nontrivial solutions to nonlinear fractional boundary value problem

$$_R D^{\alpha}_{0+} u(t) + f(t, u(t)) = 0, \quad t \in [0, 1],\ n-1 < \alpha \le n,\ n \in \mathbb{N} \tag{1}$$

$$u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \tag{2}$$

where $n \ge 2, a_i > 0$ $(i = 1, 2, \ldots, m-2)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1 f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. Also we consider the analogous problem using the Caputo fractional derivative:

$$_C D^{\alpha}_{0+} u(t) + f(t, u(t)) = 0, \quad t \in [0, 1],\ n-1 < \alpha \le n,\ n \in \mathbb{N} \tag{3}$$

$$u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i). \tag{4}$$

The paper is organized as follows. In Section 2, we present two lemmas that will be used to prove the main results. In Section 3, we obtain some existence and uniqueness results for nontrivial solution of the Riemann–Liouville multi-point boundary value problem (RLMBVP) (1)–(2) and the Caputo multi-point boundary value problem (CMBVP) (3)–(4).

2. Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the literature [2].

Definition 1 ([2]). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

provided the integral exists.

Definition 2 ([2]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : [0, \infty) \to R$ is defined to be:

$$_R D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, \quad n = [\alpha] + 1.$$

Definition 3 ([2]). The fractional derivative of a function f in the Caputo sense is defined as

$$_C D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, \quad n = [\alpha] + 1.$$

Some difference between the Riemann–Liouville fractional derivatives and Caputo fractional derivatives are highlighted by the following results:

Lemma 1 ([2]). *Let $n-1 < \alpha \le n$, $_R D^{\alpha}_{0+} u(t)$ exists for $t \in (0, 1)$. Then*

$$I^{\alpha}_{0+\,R} D^{\alpha}_{0+} u(t) = u(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - \cdots - c_n t^{\alpha-n}.$$

Lemma 2 ([2]). *Let $n-1 < \alpha \le n$, $u \in C^n[a, b]$. Then*

$$I^{\alpha}_{a\ C} D^{\alpha}_{a+} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{y^k(a)}{k!} (t-a)^k.$$

Now, let $E = C[0, 1]$, with supremum norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$ for any $y \in E$. A solution u of the BVP is called nontrivial solution if $u(t) \not\equiv 0$. In arriving at our results, we need to state two preliminary results.

Lemma 3. *Given $y \in C[0, 1]$, and $n - 1 < \alpha \le n$, the unique solution of*

$$ {}_RD^{\alpha}_{0+}u(t) + y(t) = 0, \quad 0 < t < 1, \tag{5}$$

$$ u(0) = 0, \qquad u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i) \tag{6}$$

is

$$ u(t) = \frac{1}{M_1}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i t^{\alpha-1}}{M_1}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s,$$

where $M_1 = 1 - \sum_{i=1}^{m-2} a_i {\eta_i}^{\alpha-1} \not\equiv 0$.

Proof. Assume that $u(t)$ satisfies (5), then Lemma 1 implies that

$$ u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} - I^{\alpha}_{0+} y(t). \tag{7}$$

From (6), we obtain

$$ c_2 = c_3 = \cdots = c_n = 0, \tag{8}$$

and

$$ c_1 = \frac{1}{M_1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i}{M_1}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s. \tag{9}$$

Substituting (8) and (9) in (7), one has

$$ u(t) = \frac{1}{M_1}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i}{M_1}\int_0^{\eta_i}\frac{t^{\alpha-1}(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s. \quad \square$$

Define the integral operator $T_1 : E \to E$ by

$$ \begin{aligned} T_1u(t) = {} & \frac{1}{M_1}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i}{M_1}\int_0^{\eta_i}\frac{t^{\alpha-1}(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s \\ & - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s. \end{aligned}$$

Lemma 4. *Given $y \in C[0, 1]$, and $n - 1 < \alpha \le n$, the unique solution of*

$$ {}_CD^{\alpha}_{0+}u(t) + y(t) = 0, \quad 0 < t < 1, \tag{10}$$

$$ u(0) = 0, \qquad u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \tag{11}$$

is

$$ u(t) = \frac{1}{M_2}\int_0^1 \frac{t^{n-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i t^{n-1}}{M_2}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s,$$

where $M_2 = 1 - \sum_{i=1}^{m-2} a_i {\eta_i}^{n-1} \not\equiv 0$.

Proof. Assume that $u(t)$ satisfies (10), then Lemma 2 implies that

$$ u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1} - I^{\alpha}_{0+} y(t). \tag{12}$$

From (11), we obtain

$$ c_1 = c_2 = \cdots = c_{n-1} = 0, \tag{13}$$

and

$$c_n = \frac{1}{M_2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i}{M_2}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s. \tag{14}$$

Substituting (13) and (14) in (12), one has

$$u(t) = \frac{1}{M_2}\int_0^1 \frac{t^{n-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i}{M_2}\int_0^{\eta_i}\frac{t^{n-1}(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s. \quad \square$$

Define the integral operator $T_2 : E \to E$ by

$$T_2u(t) = \frac{1}{M_2}\int_0^1 \frac{t^{n-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s - \sum_{i=1}^{m-2}\frac{a_i}{M_2}\int_0^{\eta_i}\frac{t^{n-1}(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s.$$

The RLMBVP has a solution if and only if the operator T_1 has a fixed point in E. So we only need to seek a fixed point of T_1 in E. By Ascoli–Arzela Theorem, we can prove that T_1 is a completely continuous operator. The key tool in our approach is the following Leray-Schauder nonlinear alternative.

Lemma 5 ([16]). *Let X be a real Banach space and Ω be a bounded open subset of X, $0 \in \Omega$, $F : \overline{\Omega} \longrightarrow X$ be a completely continuous operator. Then either there exist $x \in \partial\Omega$, $\lambda > 1$ such that $F(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3. Main results

For convenience, we introduce the following notation. Let

$$D_1 = \frac{1+|M_1|}{|M|_1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}k(s)\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}k(s)\mathrm{d}s,$$

$$D_2 = \frac{1+|M_2|}{|M_2|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}k(s)\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_2|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}k(s)\mathrm{d}s,$$

$$F_1 = \frac{1+|M_1|}{|M|_1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s,$$

$$F_2 = \frac{1+|M_2|}{|M_2|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_2|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s.$$

Our main result is stated as follows.

Theorem 1. *Suppose that $f(t,0) \not\equiv 0$, $t \in [a,b]$, $M \neq 0$ and there exist nonnegative functions $k, h, \in L^1[a,b]$ such that*

$$|f(t,u)| \le k(t)|u| + h(t), \quad a.\,e.(t,u) \in [a,b]\times\mathbb{R}, \tag{15}$$

$D_1 < 1$. Then RLMBVP has at least one nontrivial solution $u^{\in}C[0,1]$.

Proof. By hypothesis $D_1 < 1$. Since $f(t,0) \not\equiv 0$, there exists an interval $[\sigma,\tau] \subset [0,1]$ such that $\min_{\sigma\le t\le\tau}|f(t,0)| > 0$. On the other hand, from $h(t) \ge |f(t,0)|$, a.e. $t \in [a,b]$, we know that $F_1 > 0$. Let $m = F_1(1-D_1)^{-1}$, $\Omega = \{u \in C[0,1] : \|u\| < m\}$. Suppose $u \in \partial\Omega$, $\lambda > 1$ such that $T_1u = \lambda u$, then

$$\begin{aligned}
\lambda m = \lambda\|u\| = \|T_1u\| &= \max_{0\le t\le 1}|(T_1u)(t)| \\
&\le \frac{1}{|M_1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s, \\
&\le \frac{1+|M_1|}{|M_1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s \\
&\le \frac{1+|M_1|}{|M_1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}k(s)|u(s)|\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}k(s)|u(s)|\mathrm{d}s \\
&\quad + \frac{1+|M_1|}{|M_1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s \\
&\le D_1\|u\| + F_1 = D_1m + F_1.
\end{aligned}$$

Therefore, $\lambda \leq 1$, this contradicts $\lambda > 1$. By Lemma 5, T_1 has a fixed point in $\overline{\Omega}$. In view of $f(t,0) \not\equiv 0$, the RLMBVP has a nontrivial solution in $C[0,1]$. This completes the proof. □

Theorem 2. *Suppose that $f(t,0) \not\equiv 0$, $t \in [a,b]$, $M_2 \neq 0$ and there exist nonnegative functions $k, h, \in L^1[a,b]$ such that*

$$|f(t,u)| \leq k(t)|u| + h(t), \quad a.e.(t,u) \in [a,b] \times \mathbb{R}, \tag{16}$$

$D_2 < 1$. Then the CMBVP has at least one nontrivial solution in $C[0,1]$.

Proof. The proof of Theorem 2 is very similar to that of Theorem 1 and therefore omitted. □

Using the claimed Banach contraction principle we have:

Theorem 3. *Suppose that $f(t,0) \not\equiv 0$, $t \in [0,1]$, $M_1 \neq 0$,*

$$|f(t,u_1) - f(t,u_2)| \leq k|u_1 - u_2|,$$

a.e. $(t,u_i) \in [0,1] \times \mathbb{R}$, $(i = 1,2)$ and

$$\frac{k\left[1 + |M_1| + \sum_{i=1}^{m-2} \eta_i^{\alpha}\right]}{|M_1|\Gamma(\alpha+1)} < 1,$$

$k > 0$. Then the problem RLMPVP has a unique nontrivial solution in $\overline{\Omega}$ defined in Theorem 1.

Proof. We shall show that T_1 is a contraction.

$$\begin{aligned}
\|T_1u_1 - T_1u_2\| &= \max_{t\in[a,b]} |T_1u_1(t) - T_1u_2(t)| \\
&\leq \frac{1+|M_1|}{|M_1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u_1(s)) - f(s,u_2(s))|\mathrm{d}s \\
&\quad + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u_1(s)) - f(s,u_2(s))|\mathrm{d}s \\
&\leq \frac{1+|M_1|}{|M_1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}k\|u_1-u_2\|\mathrm{d}s + \sum_{i=1}^{m-2}\frac{a_i}{|M_1|}\int_0^{\eta_i}\frac{t^{\alpha-1}(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}k\|u_1-u_2\|\mathrm{d}s \\
&\leq \frac{k\|u_1-u_2\|[1+|M_1|+\beta\eta_i^{\alpha}]}{\Gamma(\alpha+1)}.
\end{aligned}$$

Consequently T_1 is a contraction. By the Banach fixed point theorem, we conclude that T_1 has a unique solution to the problem (1) and (2). □

Theorem 4. *Suppose that $f(t,0) \not\equiv 0$, $t \in [0,1]$, $M_2 \neq 0$, $|f(t,u_1) - f(t,u_2)| \leq k|u_1 - u_2|$, a.e. $(t,u_i) \in [0,1] \times \mathbb{R}$ $(i = 1,2)$, and $\frac{k[1+|M_2|+\sum_{i=1}^{m-2}\eta_i^{\alpha}]}{|M_2|\Gamma(\alpha+1)} < 1$, $k > 0$. Then the problem has a unique nontrivial solution $u^* \in C^1[0,1]$.*

Proof. The proof of Theorem 4 is completely similar to that of Theorem 2 and therefore omitted. □

4. Examples

In this section, in order to illustrate our results, we consider some examples.

Example 1. Consider the three-point BVP

$$\begin{aligned}
&{}_RD_{0+}^{2.8}u + 0.2t|u| + t^2 = 0, \quad 0 < t < 1, \\
&u(0) = 0, \qquad u'(0) = 0, \quad u(1) = 0.5\,u\left(\frac{1}{3}\right) + 0.2u\left(\frac{1}{2}\right).
\end{aligned} \tag{17}$$

Obviously $\alpha = 2.8,\ a_1 = 0.5,\ a_2 = 0.2, \eta_1 = \frac{1}{3},\ \eta_2 = \frac{1}{2}, n = 3$ and

$$\begin{aligned}
&f(t,x) = 0.2t|x| + t^2, \\
&k(t) = t, \quad h(t) = t^2.
\end{aligned}$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$f(t,x) \leq k(t)|x| + h(t), \quad (t,x) \in [0,1] \times \mathbb{R},$$

and $D_1 = 0.121665 < 1$. Hence, by Theorem 1, the RLMBVP (17) has at least one nontrivial solution in $C[0,1]$.

Example 2. Consider the three-point BVP

$$\begin{aligned} &{}_{C}D_{0+}^{6.7}u+\frac{ut\sin t}{t^2+1}+\mathrm{e}^t=0, \quad 0<t<1,\\ &u(0)=u'(0)=u''(0)=u'''(0)=u''''(0)=u'''''(0)=0, \quad u(1)=3u\left(\frac{1}{4}\right), \end{aligned} \tag{18}$$

where $\alpha=6.7$, $a_1=3$, $a_2=2$, $a_3=0.1$, $\eta_1=\frac{1}{4}$, $\eta_1=\frac{1}{2}$, $\eta_3=\frac{3}{4}$, $n=7$, and

$$\begin{aligned} &f(t,x)=\frac{xt\sin t}{t^2+1}+\mathrm{e}^t,\\ &k(t)=\frac{t}{t^2+1}, \quad h(t)=\mathrm{e}^t. \end{aligned}$$

It is easy to prove that $k, h\in L^1[0,1]$ are nonnegative functions and

$$f(t,x)\le k(t)|x|+h(t), \quad (t,x)\in[0,1]\times\mathbb{R}$$

and $D_2=0.0000972477<1$. Hence, by Theorem 2, the CMBVP (18) has at least one nontrivial solution in $C[0,1]$.

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